

An isothermal model of a hybrid Stirling/reverse-Brayton cryocooler

G.F. Nellis^{a,*}, J.R. Maddocks^{b,1}

^a *Cryogenic Engineering Laboratory, University of Wisconsin, Room 1339 Engineering Research Building, 1500 Engineering Drive, Madison, WI 53706, USA*

^b *Atlas Scientific 1367 Camino Robles Way, San Jose, CA 95120, USA*

Received 29 May 2002; accepted 25 October 2002

Abstract

This paper presents a model of a cryogenic refrigerator that integrates a reverse-Brayton lower temperature stage with a 2-piston Stirling upper temperature stage using a rectification system of check valves and buffer volumes. The numerical model extends the isothermal Schmidt analysis of the Stirling cycle by deriving the additional dimensionless governing equations that characterize the recuperative system. Numerical errors are quantified and the results are verified against analytical solutions in the appropriate limits. The model is used to explore the effect of the rectification system's characteristics on the overall cycle's behavior. Finally, the model is used to optimize the hybrid system's design by varying the swept volume ratio and phase angle in order to maximize the refrigeration per unit of heat transfer in the recuperator and regenerator.

© 2003 Elsevier Science Ltd. All rights reserved.

Keywords: Cryocooler; Hybrid; Stirling; Reverse-brayton; Model; Turbine; Schmidt; Isothermal; Recuperative; Regenerative

1. Introduction

This paper presents an ideal model of a cryogenic refrigeration cycle in which a recuperative lower stage is interfaced with a regenerative upper stage through a system of check valves and buffer volumes. The resulting cryocooler has a number of potential advantages. A low temperature recuperative stage avoids the dual loss mechanisms that plague regenerative cycles: thermal saturation and void volume pressurization. In addition to the anticipated higher efficiency, the configuration can be easily thermally integrated with refrigeration loads that are distributed in space or temperature. Mechanical and electrical integration is also easy due to the continuous flow nature of the recuperative cycle.

There are a number of applications that might utilize this generic concept of rectifying the AC flow within a regenerative system into a DC flow for a recuperative system. One example that is currently under develop-

ment is a pulse tube upper stage integrated with a turbomachine-based, reverse-Brayton lower stage, shown in Fig. 1a configured to cool an array of low temperature superconducting electronic circuits. This configuration represents an extremely attractive combination of refrigeration technologies because the recuperative system can be activated by a relatively large pressure difference from the linear compressor. This allows the use of simple hydrostatic gas bearings in the rotordynamic system for the turbine. Another possible configurations include a 2-piston Stirling cycle integrated with a Joule–Thomson low temperature stage, shown in Fig. 1b. Without a recuperative stage, the rectification system alone may represent a convenient method of integrating a pulse-tube system with a distributed or isolated thermal load.

The initial step in the development of these hybrid cryocoolers is to attain a solid understanding of the effect of the rectification system on the overall refrigeration cycle. Although separately Stirling and reverse-Brayton devices have been thoroughly studied, the exercise of integrating an oscillating-flow cycle with a direct-flow cycle is new. This paper describes an extension of the isothermal Schmidt analysis of a 2-piston Stirling cycle by the addition of a rectification system (check valves and buffer volumes) and a recuperative stage

* Corresponding author. Tel.: +1-608-265-6626; fax: +1-608-265-2316.

E-mail addresses: gfnellis@engr.wisc.edu (G.F. Nellis), akashani@atlasscientific.com (J.R. Maddocks).

¹ Tel.: +1-408-507-0906; fax: +1-408-927-7224.

Nomenclature

a	area, m ²
c_p	specific heat capacity, J/kg K
$\dot{m}$	mass flow rate, kg/s
m	mass, kg
p	pressure, N/m ²
$\dot{q}$	heat transfer rate, J/s
q	heat transfer per cycle, J
r_{cv}	fluid resistance of check valve, N s/kg m ²
R	gas constant, J/kg K
S	reduced dead volume
t	time, s
T	temperature, K
V	volume, m ³
$\dot{w}$	power, J/s
w	work transfer per cycle, J
A	dimensionless area
$\dot{M}$	dimensionless mass flow rate
M	dimensionless mass
$\dot{Q}$	dimensionless heat transfer rate
Q	dimensionless heat transfer per cycle
R	dimensionless fluid resistance
$\dot{W}$	dimensionless power
W	dimensionless work transfer per cycle
α	phase angle, rad

Greek symbols

β	ratio of compression to expansion temperature
χ	ratio of expansion to cold load temperature
δ	constant of solution

ϕ	crank angle, rad
γ	ratio of specific heat capacities
η	efficiency
κ	ratio of compression to expansion swept volume
λ	ratio of buffer to expansion swept volumes
Π	dimensionless pressure
θ	constant of solution
τ	cycle period, s

Subscripts

b	reverse-direction, check valve closed
B	buffer
c, C	compression space
Carnot	Carnot efficiency
charge	charge pressure
cv	check valve
d, D	dead volume
e, E	expansion space
error	numerical error
f	forward direction, check valve activated
HP	high-pressure buffer or check valve
L	cold load temperature
LP	low-pressure buffer or check valve
max	maximum
rec	recuperative
rg	regenerative
St	typical, closed Stirling cycle
t	turbine
T	total

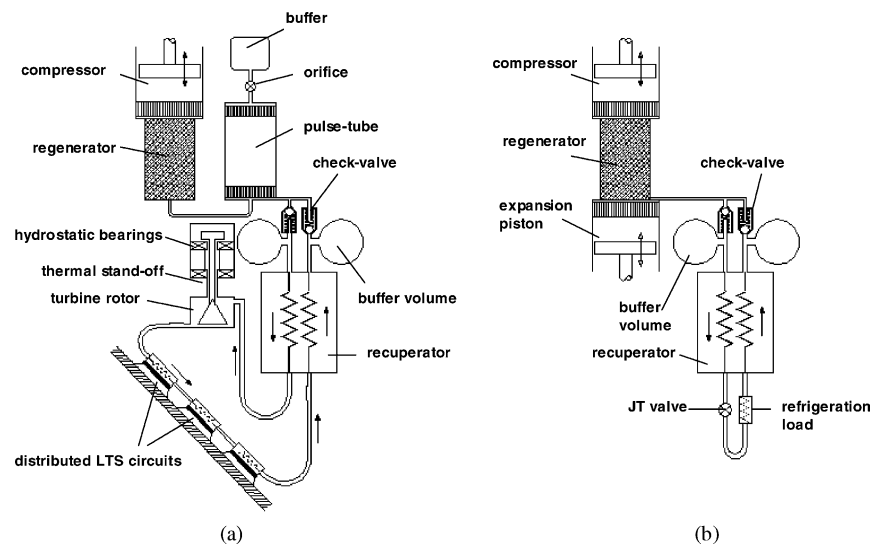


Fig. 1. Potential hybrid regenerative/recuperative cryocooler configurations: (a) pulse tube/reverse-Brayton for LTS electronics; (b) 2-piston Stirling/Joule-Thomson.

(heat exchanger and turbine). The number of governing equations is increased; reflecting the additional mass that is stored in the high- and low-pressure buffer volumes. The system of equations is solved numerically in order to determine the cyclic steady state pressure variations, which can be used to calculate various heat and work transfers. The model is presented in dimensionless form in order to identify the meaningful physical groups for this hybrid system. In addition to the typical dimensionless parameters used to characterize a Stirling system (the swept volume ratio, phase angle, reduced dead volume, and compression-to-expansion temperature ratio), the hybrid system requires dimensionless groups that characterize the rectification system (the dimensionless normal and leakage check valve resistances, and the buffer volume ratio) and the recuperative system (the dimensionless turbine size and the expansion-to-load temperature ratio).

The use of an isothermal model even in the presence of varying pressure limits the model's predictive value. However, the purpose of the model is to provide a qualitative assessment of this new type of cycle and develop an understanding the coupling between an oscillatory and continuous system. The numerical model is developed in the first section of the paper. In the second section of the paper, some important performance metrics are defined and then derived in terms of the dimensionless solution. In the third section, the numerical model is used to explore the effect of the rectification system's characteristics on the cycle's behavior. In the fourth section, an optimization technique is described in which the phase angle and swept volume ratio are varied in order to maximize the lower stage refrigeration per unit of heat transfer in the recuperator and regenerator. The results of this optimization are shown to agree with prior studies of closed Stirling systems in the appropriate limit. The fifth and last section summarizes some of the important results of this work.

2. Development of the model

Fig. 2 illustrates a 2-piston Stirling refrigerator with isothermal compression and expansion volumes and an ideal regenerator. A first-order model of this device was initially developed by Schmidt [1] and subsequently restated and expanded in many textbooks on Stirling engines and refrigerators, for example Walker [2] or Urieli and Berchowitz [3]. This analysis is an appropriate starting point for an investigation of a hybrid configuration because the characteristics of the Schmidt model are well understood allowing the effect of the additional components related to the hybrid configuration to be readily identified. The first order analysis also provides a solid foundation for more detailed analyses that will follow. For example, the assumption of isothermal

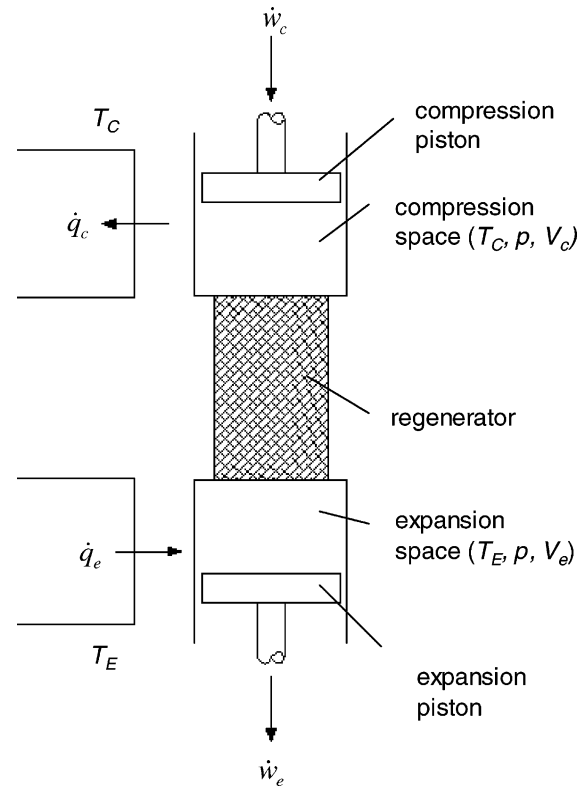


Fig. 2. 2-piston Stirling cryocooler.

compression and expansion will be relaxed to attain a more accurate, adiabatic model similar to that used by Finkelstein et al. [4]. The adaptation of the expansion process from a 2-piston mechanism to a piston-displacer or even a pulse tube configuration can also be achieved. Imperfect regeneration and recuperation can be included through the use of an ineffectiveness based on a total shuttled mass, as in Tailor and Narayankhedkar [5], or through a more sophisticated numerical solution of the heat equation within the regenerator matrix as it is subjected to the mass flow variations required by the refrigeration cycle. These first order results can be subsequently corrected for various parasitic losses such as pressure drop, shuttle heat loss, etc. which are post-calculated separately and applied to the net results, as described by Smith and co-workers (for example in [6]) and implemented by many others to achieve good agreement between analysis and data [7].

Fig. 3 illustrates the hybrid cooler that is formed by attaching a recuperative stage to the Stirling system through a rectification system that consists of two check valves, oriented in opposite directions, and two buffer volumes. The temperature–entropy diagram for the process undergone by the reverse-Brayton stage is illustrated in Fig. 4. Notice that the recuperative heat exchanger and turbine are both assumed to be ideal devices. This implies that the turbine inlet temperature is equal to the load heat exchanger exit temperature. The

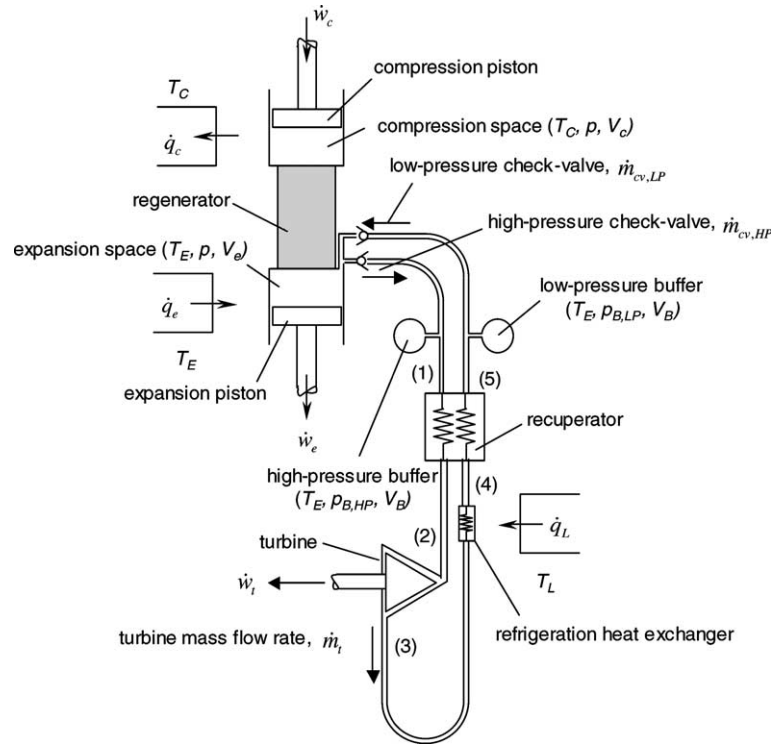


Fig. 3. Hybrid 2-piston Stirling/reverse-Brayton cryocooler.

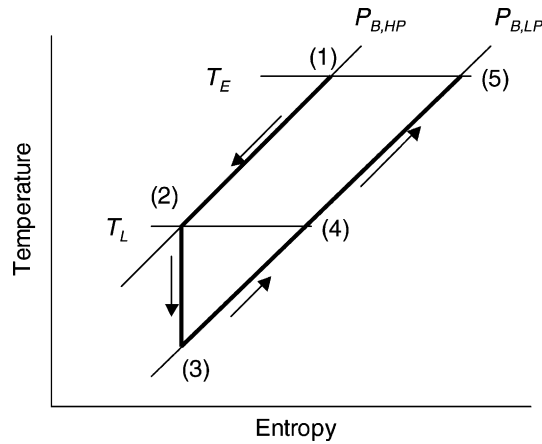


Fig. 4. Temperature-entropy diagram for reverse-Brayton stage.

only loss in the reverse-Brayton stage, when considered alone, is related to heat transfer through a temperature gradient within the refrigeration load heat exchanger. The regenerative stage, as modeled, is reversible due to the assumed perfect components and isothermal heat exchange processes. Any additional losses in the system can therefore be attributed to the rectification system, in the form of pressure drop through the check valves.

The analysis presented here will conserve much of the nomenclature used by Walker [2] in order to facilitate comparison of the results. The expansion volume (V_e) is assumed to vary sinusoidally according to

$$V_e(\phi) = \frac{V_E}{2} \cdot [1 + \cos(\phi)] \quad (1)$$

where V_E is the swept volume in the expansion space and ϕ is the crank angle

$$\phi \equiv \frac{2 \cdot \pi \cdot t}{\tau} \quad (2)$$

The volume in the compression space (V_c) is given by

$$V_c(\phi) = \frac{\kappa \cdot V_E}{2} \cdot [1 + \cos(\phi - \alpha)] \quad (3)$$

where κ is the ratio of the swept volume in the compression space to that in the expansion space and α is the phase angle. In addition, there is some unswept, or dead, volume (V_D) in the regenerative system. The total mass contained in the regenerative system (m_{rg}) can be expressed as the sum of the mass contained in these three volumes.

$$\begin{aligned} m_{rg} &= \frac{p}{R} \cdot \left(\frac{V_e}{T_E} + \frac{V_c}{T_C} + \frac{V_D}{T_D} \right) \\ &= \frac{p \cdot V_E}{2 \cdot R \cdot T_C} \cdot \{ \beta \cdot [1 + \cos(\phi)] \\ &\quad + \kappa \cdot [1 + \cos(\phi - \alpha)] + 2 \cdot S \} \end{aligned} \quad (4)$$

where S is a dimensionless dead volume (sometimes called a reduced dead volume), and β is the compression-to-expansion temperature ratio.

$$S \equiv \frac{V_D \cdot T_C}{V_E \cdot T_D}, \quad (5)$$

$$\beta \equiv \frac{T_C}{T_E} \quad (6)$$

Eq. (4) assumes that the temperatures of the expansion, compression, and dead volumes (T_E , T_C , and T_D , respectively) are constant, the instantaneous pressure in the system (p) is spatially uniform, and the working fluid is an ideal gas with a gas constant R .

In a typical closed, Stirling system (denoted by the subscript St), the observation that the total mass remains constant allows Eq. (4) to be rearranged so that pressure can be expressed as a function of crank angle

$$p_{\text{St}}(\phi) = \frac{2 \cdot R \cdot T_C \cdot m_{\text{rg}}}{V_E \cdot \{\beta \cdot [1 + \cos(\phi)] + \kappa \cdot [1 + \cos(\phi - \alpha)] + 2 \cdot S\}} \quad (7)$$

Eq. (7) can be further simplified to [2]

$$p_{\text{St}}(\phi) = \frac{2 \cdot R \cdot T_C \cdot m_{\text{rg}}}{V_E \cdot (\beta + \kappa + 2 \cdot S) \cdot [1 + \delta \cdot \cos(\phi - \theta)]} \quad (8)$$

where θ and δ are solution constants

$$\theta = \tan^{-1} \left[\frac{\kappa \cdot \sin(\alpha)}{\beta + \kappa \cdot \cos(\alpha)} \right] \quad (9)$$

$$\delta = \frac{\sqrt{\beta^2 + 2 \cdot \beta \cdot \kappa \cdot \cos(\alpha) + \kappa^2}}{\beta + \kappa + 2 \cdot S} \quad (10)$$

In the hybrid system, the mass of gas contained in the regenerative portion of the system is not constant and therefore the continuity equation for this volume must include mass flow to and from the recuperative stage

$$\frac{dm_{\text{rg}}}{dt} = \dot{m}_{\text{cv,LP}} - \dot{m}_{\text{cv,HP}} \quad (11)$$

where $\dot{m}_{\text{cv,HP}}$ and $\dot{m}_{\text{cv,LP}}$ are the mass flow rates through the high- and low-pressure check valves, respectively. Note that either of these flows may be negative in the event that the check valves leak when exposed to an adverse pressure gradient. Substituting Eq. (4) into Eq. (11) leads to

$$\begin{aligned} & \{\beta \cdot [1 + \cos(\phi)] + \kappa \cdot [1 + \cos(\phi - \alpha)] + 2 \cdot S\} \cdot \frac{dp}{d\phi} \\ & + p \cdot \{-\beta \cdot \sin(\phi) - \kappa \cdot \sin(\phi - \alpha)\} \\ & = \frac{\tau \cdot R \cdot T_C}{\pi \cdot V_E} (\dot{m}_{\text{cv,LP}} - \dot{m}_{\text{cv,HP}}) \end{aligned} \quad (12)$$

The pressures in the high- and low-pressure buffers ($p_{\text{B,HP}}$ and $p_{\text{B,LP}}$) continuously vary in response to mass flows from the regenerative system and through the turbine. Mass conservation for these buffer volumes is expressed by

$$\frac{dm_{\text{B,HP}}}{dt} = \dot{m}_{\text{cv,HP}} - \dot{m}_t, \quad (13)$$

$$\frac{dm_{\text{B,LP}}}{dt} = \dot{m}_t - \dot{m}_{\text{cv,LP}} \quad (14)$$

where $m_{\text{B,HP}}$ and $m_{\text{B,LP}}$ are the mass of gas contained in the high- and low-pressure buffer volumes, and $\dot{m}_t$ is the mass flow rate through the turbine. The buffer volumes are assumed to be at a constant temperature (T_E , the temperature of the expansion space) allowing Eqs. 13 and 14 to be rewritten as

$$\frac{dp_{\text{B,HP}}}{d\phi} = \frac{R \cdot T_E \cdot \tau}{V_B \cdot 2 \cdot \pi} \cdot (\dot{m}_{\text{cv,HP}} - \dot{m}_t), \quad (15)$$

$$\frac{dp_{\text{B,LP}}}{d\phi} = \frac{R \cdot T_E \cdot \tau}{V_B \cdot 2 \cdot \pi} \cdot (\dot{m}_t - \dot{m}_{\text{cv,LP}}) \quad (16)$$

where V_B is the volume of each buffer.

The mass flow rates in Eqs. (12), (15), and (16), must be expressed in terms of system pressures in order to solve the system of equations. A linear relationship between mass flow through and pressure drop across each check valve is assumed

$$\dot{m}_{\text{cv,HP}} = \frac{p - p_{\text{B,HP}}}{r_{\text{cv,HP}}}, \quad (17)$$

$$\dot{m}_{\text{cv,LP}} = \frac{p_{\text{B,LP}} - p}{r_{\text{cv,LP}}} \quad (18)$$

where $r_{\text{cv,HP}}$ and $r_{\text{cv,LP}}$ are the fluid resistance of the high- and low-pressure check valves. These resistances depend on the direction of the pressure difference

$$r_{\text{cv,HP}} = \begin{cases} r_{\text{cv,f}} & p > p_{\text{B,HP}} \\ r_{\text{cv,b}} & p \leq p_{\text{B,HP}} \end{cases}, \quad (19)$$

$$r_{\text{cv,LP}} = \begin{cases} r_{\text{cv,f}} & p_{\text{B,LP}} > p \\ r_{\text{cv,b}} & p_{\text{B,LP}} \leq p \end{cases} \quad (20)$$

where $r_{\text{cv,f}}$ is the forward resistance of the check valves and $r_{\text{cv,b}}$ is the resistance to back-leakage. The flow characteristic of a turbine can be approximated as a nozzle over a wide range of operating conditions [8], allowing the turbine mass flow rate to be expressed as

$$\begin{aligned} \dot{m}_t &= \left(\frac{p_{\text{B,HP}}}{R \cdot T_L} \right) \cdot a_t \\ &\cdot \sqrt{c_p \cdot 2 \cdot T_L \cdot \left[1 - \left(\frac{p_{\text{B,HP}}}{p_{\text{B,LP}}} \right)^{1-\gamma/\gamma} \right]} \end{aligned} \quad (21)$$

where a_t is an effective flow area that characterizes the turbine, T_L is the cold load temperature, c_p is the specific heat capacity at constant pressure, and γ is the ratio of specific heat capacities.

Substituting the expressions for mass flow rates given by Eqs. (17), (18) and (21) into the continuity equations in the system, Eqs. (12), (15) and (16), it is possible to obtain a system of three coupled, ordinary differential equations for the pressures within the system

$$\frac{dp}{d\phi} = \frac{p \cdot \left[\frac{T_c}{T_E} \cdot \sin(\phi) + \frac{V_c}{V_E} \cdot \sin(\phi - \alpha) \right] + \frac{\tau \cdot R \cdot T_c}{\pi \cdot V_E} \cdot \left[\frac{(p_{B,LP} - p)}{r_{cv,LP}} - \frac{(p - p_{B,HP})}{r_{cv,HP}} \right]}{\frac{T_c}{T_E} \cdot [1 + \cos(\phi)] + \frac{V_c}{V_E} \cdot [1 + \cos(\phi - \alpha)] + 2 \cdot \frac{V_D \cdot T_c}{V_E \cdot T_D}} \quad (22)$$

$$\begin{aligned} \frac{dp_{B,HP}}{d\phi} = & \frac{R \cdot T_E \cdot \tau}{2 \cdot \pi \cdot V_B} \cdot \left\{ \frac{(p - p_{B,HP})}{r_{cv,HP}} \right. \\ & \left. - \frac{p_{B,HP}}{R \cdot T_L} \cdot a_t \cdot \sqrt{2 \cdot c_P \cdot T_L \cdot \left[1 - \left(\frac{p_{B,HP}}{p_{B,LP}} \right)^{1-\gamma/\gamma} \right]} \right\} \end{aligned} \quad (23)$$

$$\begin{aligned} \frac{dp_{B,LP}}{d\phi} = & \frac{R \cdot T_E \cdot \tau}{2 \cdot \pi \cdot V_B} \\ & \cdot \left\{ \frac{p_{B,HP}}{R \cdot T_L} \cdot a_t \cdot \sqrt{2 \cdot c_P \cdot T_L \cdot \left[1 - \left(\frac{p_{B,HP}}{p_{B,LP}} \right)^{1-\gamma/\gamma} \right]} \right. \\ & \left. - \frac{(p_{B,LP} - p)}{r_{cv,LP}} \right\} \end{aligned} \quad (24)$$

Eqs. (22)–(24) suggest a set of dimensionless groups to characterize the hybrid system. The dimensionless pressure (Π), check-valve resistance (R_{cv}), turbine flow area (A_t), buffer volume (λ), and cold load temperature (χ) are defined:

$$\Pi \equiv \frac{p}{p_{\text{charge}}} \quad (25)$$

$$R_{cv} \equiv \frac{r_{cv} \cdot V_E}{R \cdot T_E \cdot \tau} \quad (26)$$

$$A_t \equiv \frac{a_t \cdot \tau}{V_E} \cdot \sqrt{\gamma \cdot R \cdot T_L} \quad (27)$$

$$\lambda \equiv \frac{V_B}{V_E} \quad (28)$$

$$\chi \equiv \frac{T_E}{T_L} \quad (29)$$

where p_{charge} is the charge pressure.

The dimensionless form of Eqs. (22)–(24) are

$$\frac{d\Pi}{d\phi} = \frac{\frac{\beta}{\pi} \cdot \left[\frac{(\Pi_{B,LP} - \Pi)}{R_{cv,LP}} - \frac{(\Pi - \Pi_{B,HP})}{R_{cv,HP}} \right] + \Pi \cdot [\beta \cdot \sin(\phi) + \kappa \cdot \sin(\phi - \alpha)]}{\{\beta \cdot [1 + \cos(\phi)] + \kappa \cdot [1 + \cos(\phi - \alpha)] + 2 \cdot S\}} \quad (30)$$

$$\begin{aligned} \frac{d\Pi_{B,HP}}{d\phi} = & \frac{(\Pi - \Pi_{B,HP})}{2 \cdot \pi \cdot R_{cv,HP} \cdot \lambda} - \frac{A_t \cdot \Pi_{B,HP} \cdot \chi}{2 \cdot \pi \cdot \lambda} \\ & \cdot \sqrt{\frac{2}{(\gamma - 1)} \cdot \left[1 - \left(\frac{\Pi_{B,HP}}{\Pi_{B,LP}} \right)^{1-\gamma/\gamma} \right]} \end{aligned} \quad (31)$$

$$\begin{aligned} \frac{d\Pi_{B,LP}}{d\phi} = & -\frac{(\Pi_{B,LP} - \Pi)}{2 \cdot \pi \cdot R_{cv,LP} \cdot \lambda} + \frac{A_t \cdot \Pi_{B,HP} \cdot \chi}{2 \cdot \pi \cdot \lambda} \\ & \cdot \sqrt{\frac{2}{(\gamma - 1)} \cdot \left[1 - \left(\frac{\Pi_{B,HP}}{\Pi_{B,LP}} \right)^{1-\gamma/\gamma} \right]} \end{aligned} \quad (32)$$

These ordinary differential equations are stepped forward in time using a Runge–Kutta fourth-order technique starting from an initial charge pressure ($\Pi = \Pi_{B,HP} = \Pi_{B,LP} = 1$) until a cyclic steady state is achieved. Cyclic steady-state is defined as the condition where the maximum change in any of the dimensionless pressures is less than $1e-4$ between its value at the start and end of a cycle. The mathematical model is implemented in the EES—Engineering Equation Solver [9] software.

3. Performance metrics

The result of the model described in the previous section is the prediction of the cyclic steady-state pressure variation in the regenerative cycle and the two buffer volumes. In this section, we reduce these pressure variations into some important energy transfers that characterize the system.

In order to compute the performance of the regenerative stage, the heat transfer into the expansion space (q_E) must be computed. Provided both the regenerator and recuperator are perfect, this heat transfer is equivalent to the work transfer from the expansion space (w_E).

$$q_E = w_E = \int_0^{2\pi} p(\phi) \cdot \frac{dV_c}{d\phi} d\phi \quad (33)$$

These and all subsequently considered work and heat transfers are made dimensionless (W and Q) through comparison with the product of the charge pressure and the expansion volume

$$\begin{aligned} Q_E = W_E & \equiv \frac{w_E}{p_{\text{charge}} \cdot V_E} \\ & = -\frac{1}{2} \cdot \int_0^{2\pi} \Pi(\phi) \cdot \sin(\phi) d\phi \end{aligned} \quad (34)$$

The power consumption of the refrigerator is related to the work transfer into the compression space (w_C). This is non-dimensionalized (W_C) and computed in a similar manner.

$$W_C \equiv \frac{w_C}{p_{\text{charge}} \cdot V_E} = \frac{1}{2} \cdot \kappa \cdot \int_0^{2\pi} \Pi(\phi) \cdot \sin(\phi - \alpha) d\phi \quad (35)$$

The performance of the recuperative stage is related to the heat transfer into the refrigeration load heat exchanger (q_L). This heat transfer is equal to the work transfer from the turbine (w_t) which is, in turn, related to the mass flow rate through the recuperative stage. The instantaneous, dimensionless mass flow rate through the recuperative stage ($\dot{M}_t$) is

$$\begin{aligned} \dot{M}_t &\equiv \frac{\dot{m}_t \cdot \tau \cdot R \cdot T_C}{p_{\text{charge}} \cdot V_E} \\ &= \beta \cdot \chi \cdot A_t \cdot \Pi_{\text{B,HP}} \\ &\quad \cdot \sqrt{\frac{2}{(\gamma - 1)}} \cdot \left[1 - \left(\frac{\Pi_{\text{B,HP}}}{\Pi_{\text{B,LP}}} \right)^{1-\gamma/\gamma} \right] \end{aligned} \quad (36)$$

The total dimensionless mass processed by the turbine per cycle (M_t) is obtained by integrating Eq. (36)

$$M_t \equiv \frac{\dot{m}_t \cdot R \cdot T_C}{p_{\text{charge}} \cdot V_E} = \frac{1}{2 \cdot \pi} \cdot \int_0^{2\pi} \dot{M}_t d\phi \quad (37)$$

The instantaneous power produced by the turbine ($\dot{w}_t$) is given by

$$\dot{w}_t = \dot{m}_t \cdot c_p \cdot T_L \cdot \left[1 - \left(\frac{p_{\text{B,HP}}}{p_{\text{B,LP}}} \right)^{1-\gamma/\gamma} \right] \quad (38)$$

Eq. (38) implies that the turbine is reversible and that the working fluid is a perfect gas (i.e. an ideal gas with a constant specific heat capacity). The turbine power may be transferred to a higher temperature as mechanical work transfer in a rotating shaft and dissipated as fluid power, as described by Swift [10], or transferred as electrical power and dissipated as resistive heating, as described by Nellis et al. [11]. The dimensionless instantaneous turbine power ($\dot{W}_t$) is

$$\begin{aligned} \dot{W}_t &\equiv \frac{\dot{w}_t \cdot \tau}{V_E \cdot p_{\text{charge}}} \\ &= \frac{\dot{M}_t \cdot \gamma}{(\gamma - 1) \cdot \beta \cdot \chi} \cdot \left[1 - \left(\frac{\Pi_{\text{B,HP}}}{\Pi_{\text{B,LP}}} \right)^{1-\gamma/\gamma} \right] \end{aligned} \quad (39)$$

The total work produced by the turbine per cycle (W_t) is obtained by integrating Eq. (39):

$$W_t = \frac{w_t}{V_E \cdot p_{\text{charge}}} = \frac{1}{2 \cdot \pi} \cdot \int_0^{2\pi} \dot{W}_t d\phi \quad (40)$$

If the recuperator is perfect, then the turbine power is equal to the low temperature refrigeration load (Q_L)

$$Q_L \equiv \frac{q_L}{V_E \cdot p_{\text{charge}}} = W_t \quad (41)$$

The Carnot efficiency (η_{Carnot}) of a Stirling cycle modeled by an isothermal analysis is necessarily equal to one because no irreversible processes are included. However, some entropy generation is introduced as the reverse-Brayton cycle interacts with a constant temperature load, even in the limit of perfect recuperation and isentropic expansion. Also, the rectification system may introduce entropy generation related to the pressure drop across the check valves. The Carnot efficiency of the hybrid cryocooler is defined as the ratio of the power required by a reversible system supplying the same refrigeration loads at the same temperatures to the actual power.

$$\eta_{\text{Carnot}} = \frac{Q_L \cdot (\beta \cdot \chi - 1) + Q_E \cdot (\beta - 1)}{Q_C - Q_E - Q_L} \quad (42)$$

Finally, the heat transfers in both heat exchangers, the regenerator and recuperator (q_{rg} and q_{rec}), are of interest as these quantities determine the overall amount of heat transfer area required and, by extension, the ultimate size of the cryocooler. The total mass of gas that is shuttled through the regenerator during each cycle is equal to the maximum mass of gas contained in the compression space during a cycle. At any instant, the mass of gas in the compression space (m_C) can be calculated according to

$$m_C = \frac{V_C \cdot p}{2 \cdot R \cdot T_C} \cdot [1 + \cos(\phi - \alpha)] \quad (43)$$

This mass is made dimensionless (M_C) according to

$$M_C \equiv \frac{m_C \cdot R \cdot T_C}{p_{\text{charge}} \cdot V_E} = \frac{\Pi \cdot \kappa}{2} \cdot [1 + \cos(\phi - \alpha)] \quad (44)$$

The heat transfer per cycle in the regenerator is related to shuttling fluid from the compression space to the expansion space:

$$q_{\text{rg}} = \max(m_C) \cdot c_p \cdot (T_C - T_E) \quad (45)$$

The regenerator heat transfer is made dimensionless (Q_{rg}) according to

$$Q_{\text{rg}} \equiv \frac{q_{\text{rg}}}{p_{\text{charge}} \cdot V_E} = \frac{M_{\text{C,max}} \cdot \gamma}{(\gamma - 1)} \cdot \frac{(\beta - 1)}{\beta} \quad (46)$$

The total amount of heat transferred in the recuperator during each cycle is obtained by integrating the instantaneous rate of heat transfer

$$q_{\text{rec}} = \int_0^\tau \dot{m}_t \cdot c_p \cdot (T_E - T_L) dt \quad (47)$$

This recuperator heat transfer is made dimensionless (Q_{rec}) according to

$$Q_{\text{rec}} \equiv \frac{q_{\text{rec}}}{p_{\text{charge}} \cdot V_E} = \frac{M_t \cdot \gamma \cdot (\chi - 1)}{(\gamma - 1) \cdot \beta \cdot \chi} \quad (48)$$

If the check valves in the hybrid system are shut (i.e. if $R_{\text{cv,f}} = R_{\text{cv,b}} \rightarrow \infty$) then the results predicted by the numerical model described above agree with those

predicted by the Schmidt analysis, verifying this aspect of the numerical model. The dimensionless input power predicted by the numerical model converges to the Schmidt solution to within 0.1% in this limit when more than 20 integration steps are used. The model has also been verified by comparing the instantaneous mass in the system to the charge mass; for more than 100 integration steps this numerical error is found to be less than 0.01%. At least 100 integration steps are used throughout the remainder of this paper.

4. Effect of the rectification components

Fig. 5 illustrates the pressure waves as a function of crank angle for the hybrid cycle as well as the pressure wave predicted by Eq. (49) for the Stirling cycle limit ($R_{cv,f} = R_{cv,b} \rightarrow \infty$). The most obvious effect of opening the check valves and connecting the recuperative stage is a reduction in the amplitude of the pressure wave experienced by the regenerative stage. The crank angle at which the pressure reaches its maximum value is also slightly reduced. In the recuperative stage, the pressures oscillate more gently related to the cyclic draining and filling of the buffer volumes. The pressure difference between the high- and low-pressure buffers drives the turbine and produces refrigeration at the cold load temperature. The rectification system components are responsible for changing the oscillating pressure in the regenerative system (Π) into the (more) steady pressures in each buffer volume ($\Pi_{B,HP}$ and $\Pi_{B,LP}$). These components include the check valves and the buffer volumes and are characterized by the dimensionless groups $R_{cv,f}$, $R_{cv,b}$, and λ . The effect of these dimensionless groups is

straightforward; the optimal rectification system will have large buffer volumes compared to the flow per cycle required by the turbine ($\lambda/A_t \rightarrow \infty$), check valves that offer no resistance to flow in the on-state relative to the turbine itself ($R_{cv,f}A_t \rightarrow 0$) and infinite resistance in the off-state ($R_{cv,b} \rightarrow \infty$). Fig. 6 illustrates the pressure waves as the rectification system approaches this ideal limit, very large buffer volumes and check valves that approach perfect behavior. The regenerative system pressure oscillates between the pressure in the high- and low-pressure buffer volumes and these pressures are nearly constant. There is little pressure drop across the check valves when they are activated. The Carnot efficiency of this cycle is high because the ideal rectification system has not introduced any losses to the overall cycle. The turbine power produced by this cycle is constant—the infinite buffer volumes present it with a constant pressure ratio.

Fig. 7 illustrates the situation where the check valve resistance remains negligible but now the buffer volume is finite. The buffer pressures are no longer constant and therefore the turbine sees a time varying driving pressure. While the Carnot efficiency of the cycle in this limit is still high, the fluctuating turbine power may be problematic for a real system; depending on the inertia of the rotating assembly the speed of the turbine may vary and in any event the turbine will be forced to operate away from its design point for a significant portion of each cycle. Fig. 8 illustrates the magnitude of the cyclic variation in the turbine power normalized against its average power as a function of the buffer volume for different values of check valve resistance. At high values of buffer volume the turbine power variation is small due to the stability provided by the large amount of fluid

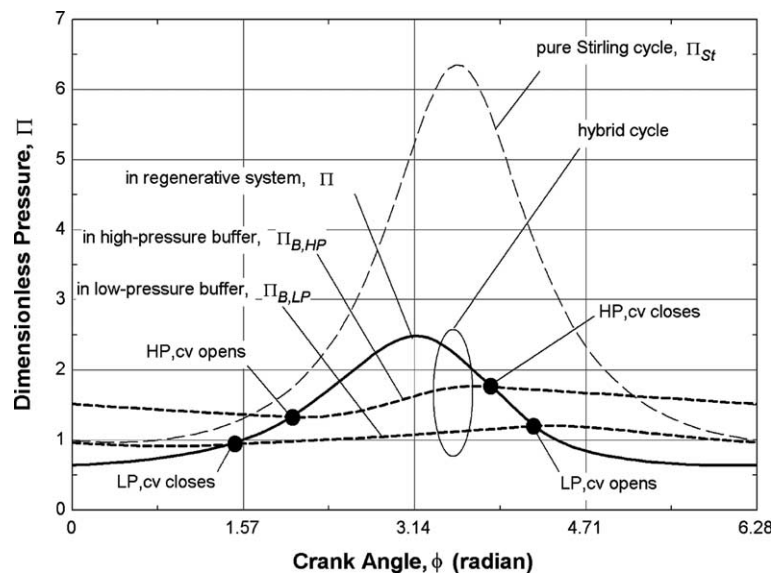


Fig. 5. Effect of rectification system on the pressure variations for $R_{cv,f} = 0.2$, $R_{cv,b} \rightarrow \infty$, $\beta = 3.0$, $\kappa = 2.0$, $S = 0.5$, $\alpha = 1.0$ rad, $\lambda = 1.25$, $\gamma = 5/3$, $\chi = 3.0$, $A_t = 0.25$.

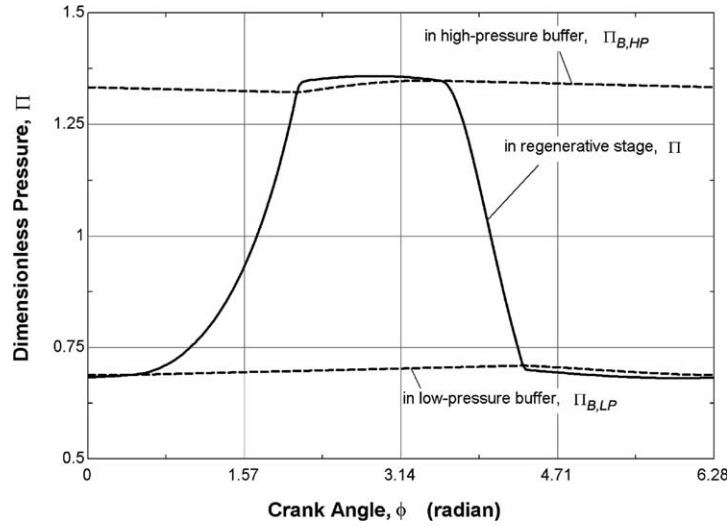


Fig. 6. Pressure waves in cycle for near perfect rectification system ($R_{cv,f}A_t$ approaching 0, $R_{cv,b}$ approaching ∞ , λ/A_t approaching ∞) for $\beta = 3.0$, $\kappa = 2.0$, $S = 0.5$, $\alpha = 1.0$ rad, $\gamma = 5/3$, $\chi = 3.0$, $A_t = 0.25$.

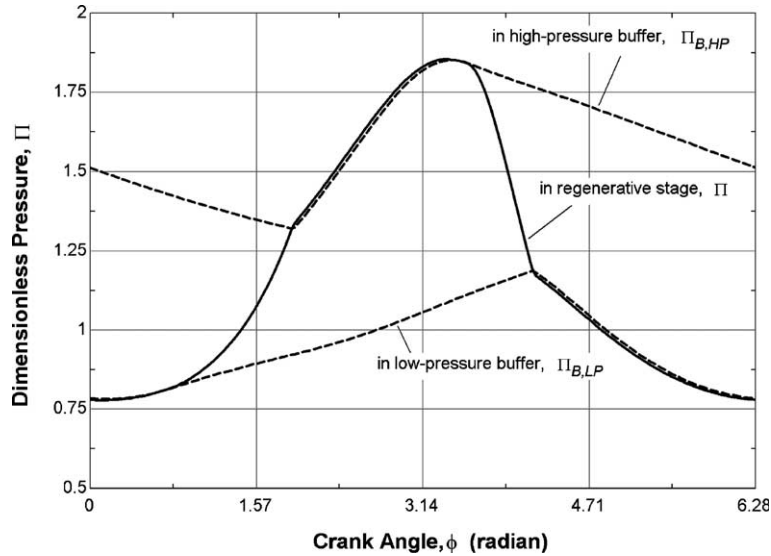


Fig. 7. Pressure waves in cycle with near perfect check valves ($R_{cv,f}A_t$ approaching 0, $R_{cv,b} \rightarrow \infty$) but finite buffer volume ($\lambda/A_t = 5.0$) for $\beta = 3.0$, $\kappa = 2.0$, $S = 0.5$, $\alpha = 1.0$ rad, $\gamma = 5/3$, $\chi = 3.0$, $A_t = 0.25$.

capacitance. As the check valve resistance is increased, the turbine tends to run in very short spurts twice per cycle that correspond to the pressure within the regenerative system hitting its maximum and minimum values.

Fig. 9 illustrates the situation where the buffer volumes are infinite but now the check valve resistance is finite. In this limit, the pressure in the buffer volume remains constant but the pressure in the regenerative stage must significantly over- and under-shoot the buffer pressures in order to provide the pressure difference required by the check valves. The net effect is to reduce the efficiency of the cycle due to this additional loss mechanism. Fig. 10 illustrates the Carnot efficiency as a

function of check valve resistance for various values of buffer volume. At a very low value of check valve resistance they introduce no losses and therefore the efficiency limits to a value dictated only by losses in the refrigeration heat exchanger. At very high values of check valve resistance, the efficiency approaches unity because there is no mass flow rate through the valves and therefore no entropy generation in the recuperative stage at all. In the intermediate region, the degradation in efficiency is dependent on the buffer volume; large buffer volumes tend to degrade the efficiency because the buffer pressures are not able to follow the pressure within the regenerative cycle and therefore the valve pressure drop is higher.

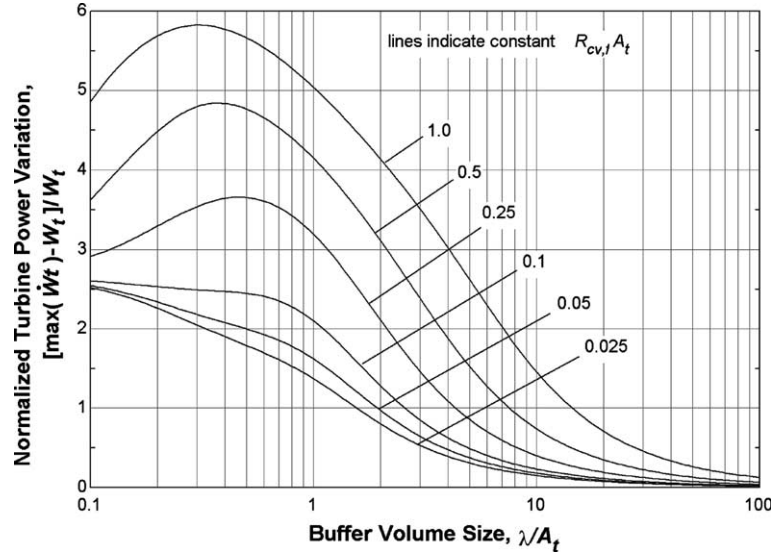


Fig. 8. Fluctuation in turbine power as a function of buffer volume at various check valve resistances for $R_{cv,b} \rightarrow \infty$, $\beta = 3.0$, $\kappa = 2.0$, $S = 0.5$, $\alpha = 1.0$ rad, $\gamma = 5/3$, $\chi = 3.0$, $A_t = 0.25$.

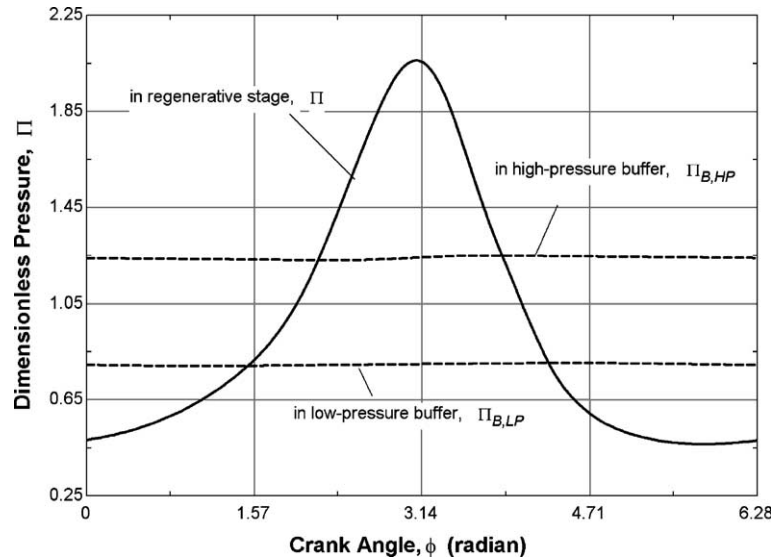


Fig. 9. Pressure waves in cycle with infinite buffer volume ($\lambda/A_t \rightarrow \infty$) but imperfect check valves ($R_{cv,f} A_t = 0.05$, $R_{cv,b} \rightarrow \infty$) for $\beta = 3.0$, $\kappa = 2.0$, $S = 0.5$, $\alpha = 1.0$ rad, $\gamma = 5/3$, $\chi = 3.0$, $A_t = 0.25$.

From the discussion above it is clear that the rectification system should possess a dimensionless buffer volume to dimensionless turbine flow area ratio, λ/A_t , that is large (in excess of 5) in order to allow the turbine to operate near its design point at all times. Also, the product of the dimensionless check valve resistance and the dimensionless turbine flow area should be small (below 0.05) so that the rectification system does not introduce significant losses.

5. Optimization of the hybrid cycle

The performance of the hybrid cooler is a function of nine dimensionless groups.

$$\begin{aligned} Q_L, Q_E, W_C, \eta_{\text{Carnot}}, Q_{\text{rec}}, Q_{\text{rg}} \\ = \mathfrak{F}(\beta, \chi, S, R_{cv,f}, R_{cv,b}, \lambda, \kappa, \alpha, A_t) \end{aligned} \quad (49)$$

The temperature ratios between the two stages (β and χ) are likely to be specified according to the requirements of a particular application. The dead volume (S), check valve resistances ($R_{cv,f}$ and $R_{cv,b}$), and buffer volume (λ) are all important to performance, but they are not design parameters in that there is a clear optimal limit for each of these groups. In a multi-stage refrigeration system, the ratio of the upper- to lower-stage refrigeration loads (Q_E/Q_L) is likely to be specified along with the temperature ratios. When all other parameters are fixed, this ratio can be adjusted directly by varying the di-

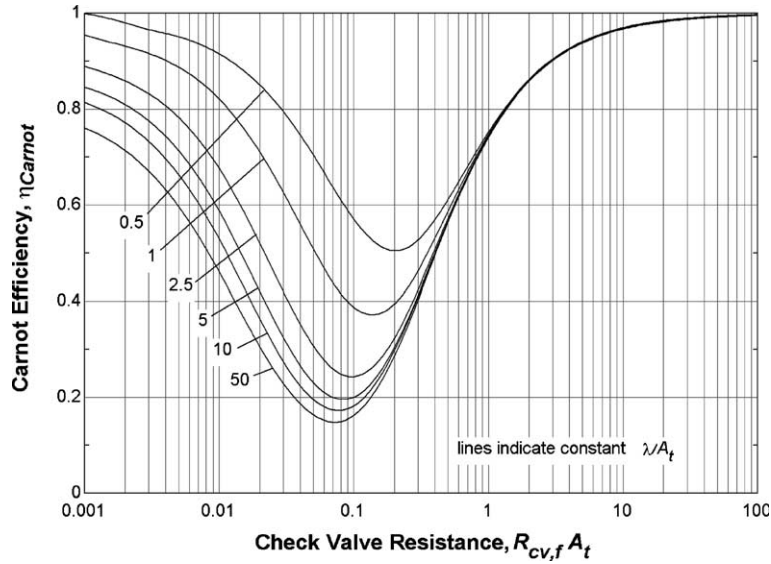


Fig. 10. Carnot efficiency as a function of check valve resistance at various values of buffer volume for $R_{cv,b} \rightarrow \infty$, $\beta = 3.0$, $\kappa = 2.0$, $S = 0.5$, $\alpha = 1.0$ rad, $\gamma = 5/3$, $\chi = 3.0$, $A_t = 0.25$.

mensionless turbine area (A_t), which allows the turbine to accept a larger or smaller amount of flow. As the dimensionless turbine area is increased, more heat is lifted in the recuperative stage and less in the regenerative stage.

This leaves two “free” parameters that can be varied to optimize the hybrid system: the swept volume ratio (κ) and the phase angle (α). The isothermal Schmidt model of the closed Stirling cycle has been well studied and Walker [2] showed that there exists an optimal phase angle and swept volume ratio that maximizes the refrigeration power per unit mass of gas in the system. This metric is less meaningful in a hybrid system because

a large part of the fluid inventory is contained in the buffer volumes and therefore does not participate in the refrigeration process. A more important parameter is the heat lifted from the recuperative stage per unit of heat transfer in the regenerator and recuperator. The total heat transfer in these heat exchange devices will be proportional to their physical size and therefore this optimization will provide the most refrigeration for a fixed amount of heat transfer area.

Fig. 11 illustrates contours of refrigeration load per heat exchanger load as a function of the swept volume ratio and the phase angle and clearly shows that there is an optimal swept volume ratio and phase angle for the

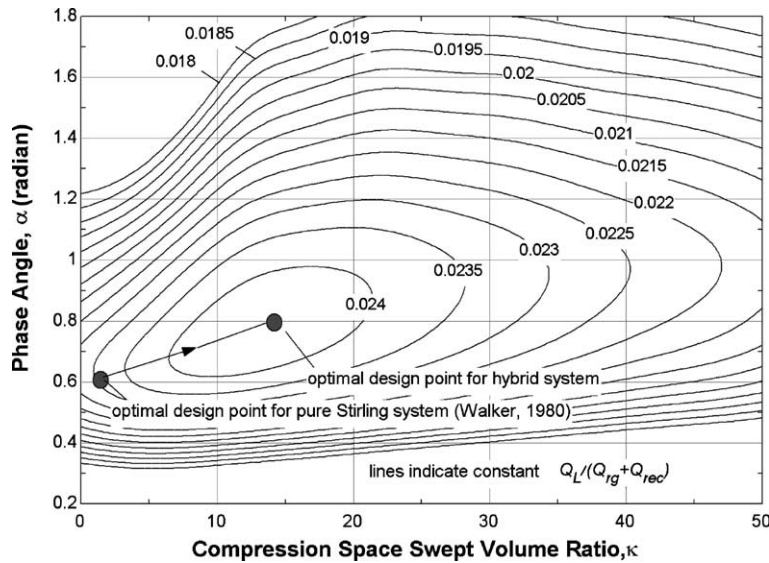


Fig. 11. Contours of constant recuperative stage refrigeration load per heat exchanger load for $\lambda/A_t = 5.0$, $R_{cv,f}A_t = 0.02$, $R_{cv,b} \rightarrow \infty$, $\beta = 3.0$, $S = 0.5$, $\gamma = 5/3$, $\chi = 3.0$, $Q_E/Q_L = 5.0$.

hybrid system that makes the best use of the available heat transfer area. Walker [2] showed that the optimal phase angle for a typical closed Stirling system lies between 0.55 and 0.65 rad over a wide range of conditions. In the hybrid system this optimal value increases to near 0.85 rad, consistent with the earlier observation that one effect of the rectification system is to move the peak of the pressure wave to an earlier point in the cycle. According to Walker [2], the optimal swept volume ratio for a closed Stirling system is proportional to the temperature ratio (β) and ranges from about $\kappa = 5$ at $\beta = 10$ to $\kappa = 1$ at $\beta = 3$. In the hybrid system the optimal swept volume ratio is much higher, consistent with the fact that the compressor now serves the dual purpose of energizing the recuperative and regenerative refrigeration stages.

Fig. 12 illustrates the optimal swept volume ratio and phase angle as a function of the ratio of the regenerative to recuperative refrigeration load. At very high values of Q_E/Q_L , the system limits to a closed Stirling cycle and the optimization process approaches the appropriate optimal values of phase angle and swept volume ratio. As more refrigeration is provided by the recuperative stage the optimal swept volume ratio increases in order to provide additional, rectified flow and the optimal phase angle rises gradually. In the opposite limit, Q_E/Q_L approaching zero, all of the refrigeration is supplied at the cold temperature and therefore the optimal swept volume ratio approaches infinity. In an analysis that accounts for recuperator and regenerator losses, a finite swept volume in the expansion space will be desirable even in this limit in order to intercept these heat exchanger losses at the expansion rather than the cold load temperature.

6. Summary

A hybrid cryogenic refrigeration cycle is described in which an oscillating flow from an upper stage, regenerative cycle is rectified and used to supply a continuous flow to a lower stage, recuperative cycle. The potential advantages of this type of cycle include avoiding low temperature regenerative losses and alleviating mechanical, thermal, and electrical integration issues. There are numerous applications that might benefit from this generic concept and therefore this paper presents a first order analysis of a hybrid system in which a 2-piston Stirling cycle with a perfect regenerator and isothermal compression and expansion spaces is integrated with a reverse-Brayton stage with a perfect recuperator and a reversible turbine through a simple rectification system that consists of check valves and buffer volumes. The model is presented as an extension of the isothermal Schmidt analysis and the numerical results are verified against this limit.

The model is used to explore the effects of the rectification system on the overall system performance and also to optimize the hybrid system for a given set of operating conditions. Some of the important results include

- The ideal rectification system has infinite buffer volumes and check valves that present no resistance to normal flow and allow no reverse leakage; in this limit the turbine produces a constant power and the rectification system introduces no losses.
- When the buffer volumes are finite, the flow through the recuperative stage oscillates. This causes a corresponding oscillation in the turbine power during a

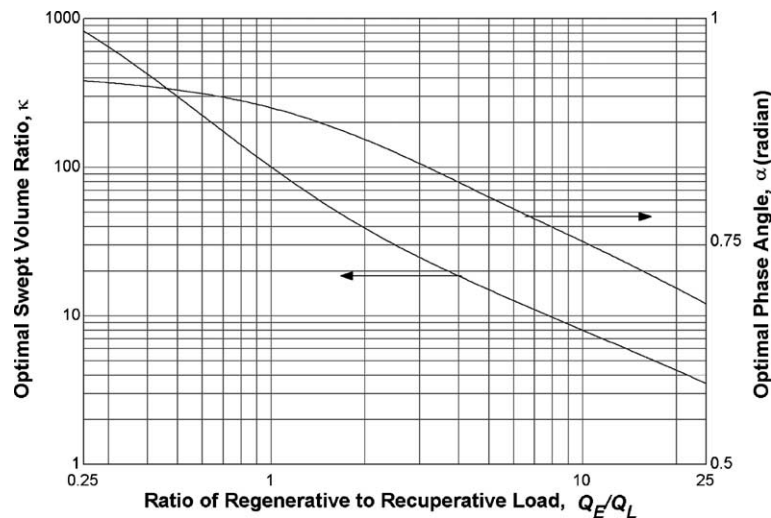


Fig. 12. Optimal values of swept volume ratio and phase angle as a function of Q_E/Q_L for $\lambda/A_t = 5.0$, $R_{cv,f}A_t = 0.02$, $R_{cv,b} \rightarrow \infty$, $\beta = 3.0$, $S = 0.5$, $\gamma = 5/3$, $\chi = 3.0$.

cycle. The magnitude of this power oscillation is quantified and presented as a function of the dimensionless buffer volume and check valve resistance.

- When the check valve resistances are finite, the pressure drop through the valves causes a reduction in the cycle efficiency. The magnitude of this loss is quantified and presented as a function of the dimensionless check valve resistance and buffer volume.
- There exists an optimal phase angle and swept volume ratio that maximizes the heat lifted from the recuperative stage per load on the heat exchangers (regenerator and recuperator). The optimal value of the phase angle is slightly higher than in a “pure” Stirling system and the optimal value of swept volume ratio is much larger. These values are presented for one particular set of operating conditions as the ratio of the heat lifted by the recuperative stage to the regenerative stage is varied.

Acknowledgement

The support of Atlas Scientific is gratefully acknowledged.

References

- [1] Schmidt G. Theore der Lehmannschen Calorischen Maschine. *Z Verb dt Ing* 1871;15(1).
- [2] Walker G. *Stirling engines*. Oxford: Clarendon Press; 1980. p. 50–8.
- [3] Urieli I, Berchowitz DM. *Stirling cycle engine analysis*. Bristol: Adam Hilger; 1984. p. 20–50.
- [4] Organ AJ. *Thermodynamics and gas dynamics of the Stirling cycle machine*. Cambridge University Press; 1992. [Appendix VII].
- [5] Tailor PR, Narayankhedkar KG. Optimum design charts for a piston-displacer Stirling cryocooler. In: Fast RW (Ed.), *Advances in cryogenic engineering*, vol. 35. New York: Plenum Press; 1990. p. 1407–14.
- [6] Rios PA, Smith JL Jr., Qvale EG. *Advances in cryogenic engineering*, vol. 14. New York: Plenum Press; 1969. p. 332.
- [7] Walker G, Fauvel OR, Reader G, Bingham ER. *The Stirling alternative: power systems, refrigerants, and heat pumps*. Switzerland: Gordon and Breach Science Publishers; 1994.
- [8] Cohen H, Rogers GFC, Saravanamuttoo HHH. *Gas turbine theory*. 3rd ed. New York: Longman Scientific & Technical; 1987.
- [9] Klein SA, Alvarado FL. EES—engineering equation solver. F-Chart Software 2002. Available from: <http://www.fchart.com>.
- [10] Swift WL. Single-stage reverse Brayton cryocooler: performance of the engineering model. In: Ross RG Jr. (Ed.), *Cryocoolers*, vol. 8. New York: Plenum Press; 1995. p. 499–506.
- [11] Nellis G, Dolan F, McCormick J, Swift W, Sixsmith H, Gibbon J, et al. Reverse Brayton cryocooler for NICMOS. In: Ross RG Jr. (Ed.), *Cryocoolers*, vol. 10. New York: Kluwer Academic/Plenum; 1999. p. 431–8.